Sensing and Observation Theory

part 2

Welcome to...

Modelling, Uncertainty, and Data for Engineers

WEEK 8 !

Course evaluation

Please help us evaluate and improve MUDE with your feedback!

Evasys (evaluation-ceg@tudelft.nl)

SCAN ME!

MUDE survey (mude-ceg@tudelft.nl)

ME TOO!

6. Observation theory

- 6.1. Introduction
- 6.2. Least-squares estimation
- 6.3. Weighted least-squares estimation
- 6.4. Best linear unbiased estimation
- 6.5. Precision and confidence intervals
- 6.6. Maximum Likelihood Estimation
- 6.7. Non-linear least-squares estimation
- 6.8. Model testing
- 6.9. Hypothesis testing for Sensing and Monitoring
- 6.10. Notation and formulas

- 6.7. Non-linear least-squares estimation
  - Notebook Gauss-Newton iteration for GNSS Trilateration
- 6.8. Model testing
- 6.9. Hypothesis testing for Sensing and Monitoring
  - Notebook exercise: which melting model is better?
  - Notebook exercises: is my null hypothesis good enough?

Review

Estimators - overview

Functional model: $\mathbb{E}(Y) = \mathrm{A} \cdot \mathrm{x}$

Stochastic model: $\mathbb{D}(Y) = \Sigma_Y$

- Weighted Least-Squares estimation : minimizing weighted sum of squared errors

  allows to give different weights to observations

  $$\hat{X} = (\mathrm{A}^T \mathrm{W} \mathrm{A})^{-1} \mathrm{A}^T \mathrm{W} \cdot Y$$

- Best Linear Unbiased estimation : $\min\left(trace(\Sigma_{\hat{X}})\right)$ (best), $\hat{X} = \mathrm{L}^T \cdot Y$ (linear), $\mathbb{E}(\hat{X}) = \mathrm{x}$ (unbiased)

  $$\hat{X} = \left(\mathrm{A}^T \Sigma_Y^{-1} \mathrm{A}\right)^{-1} \mathrm{A}^T \Sigma_Y^{-1} Y$$

- Maximum Likelihood estimation : most likely x for given y,

  for normally distributed data same as BLUE

Maximum likelihood estimation is an important method, also for machine learning (Q2), and is included as reference material (will not be on exam or in assignments)

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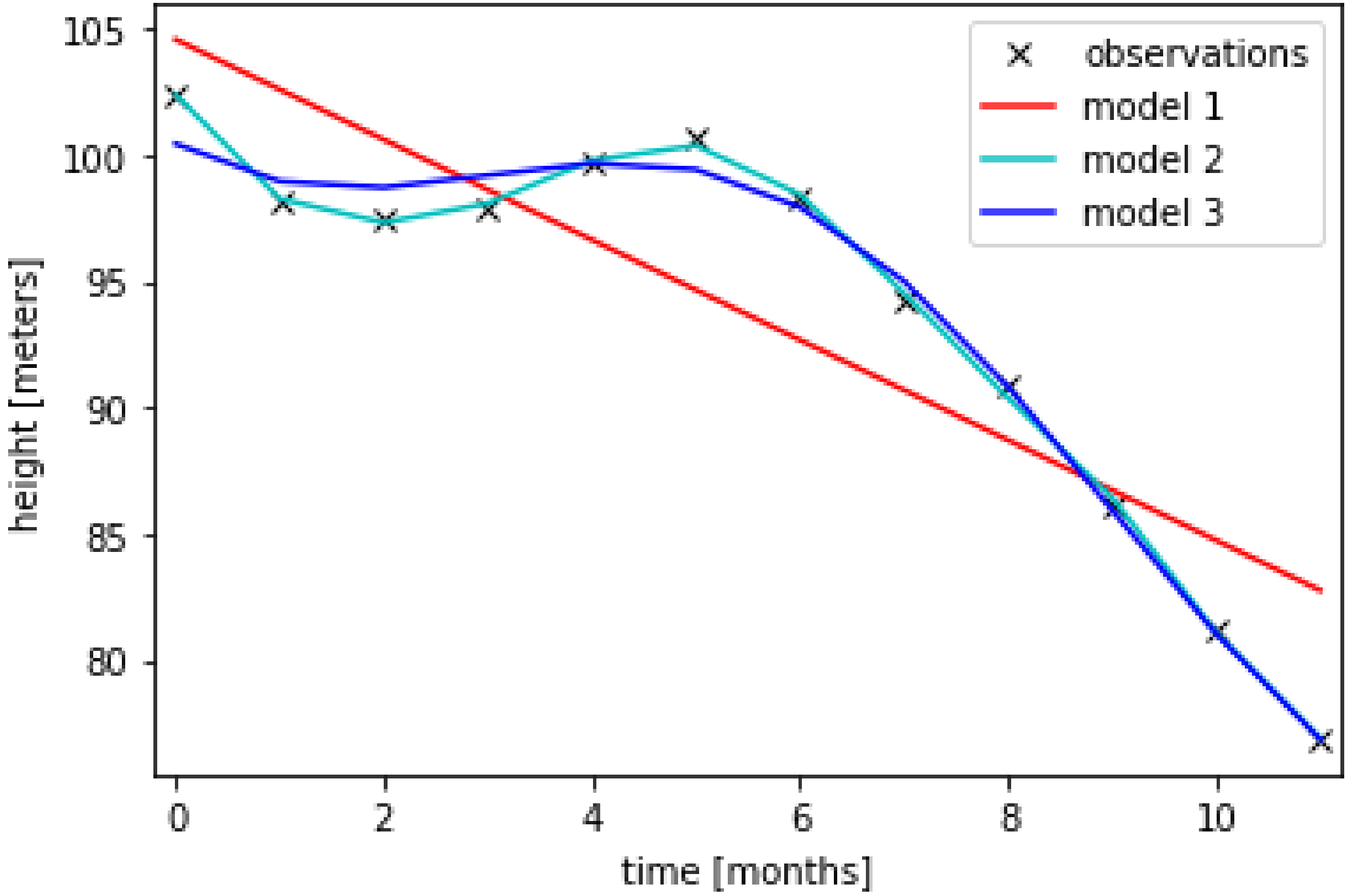
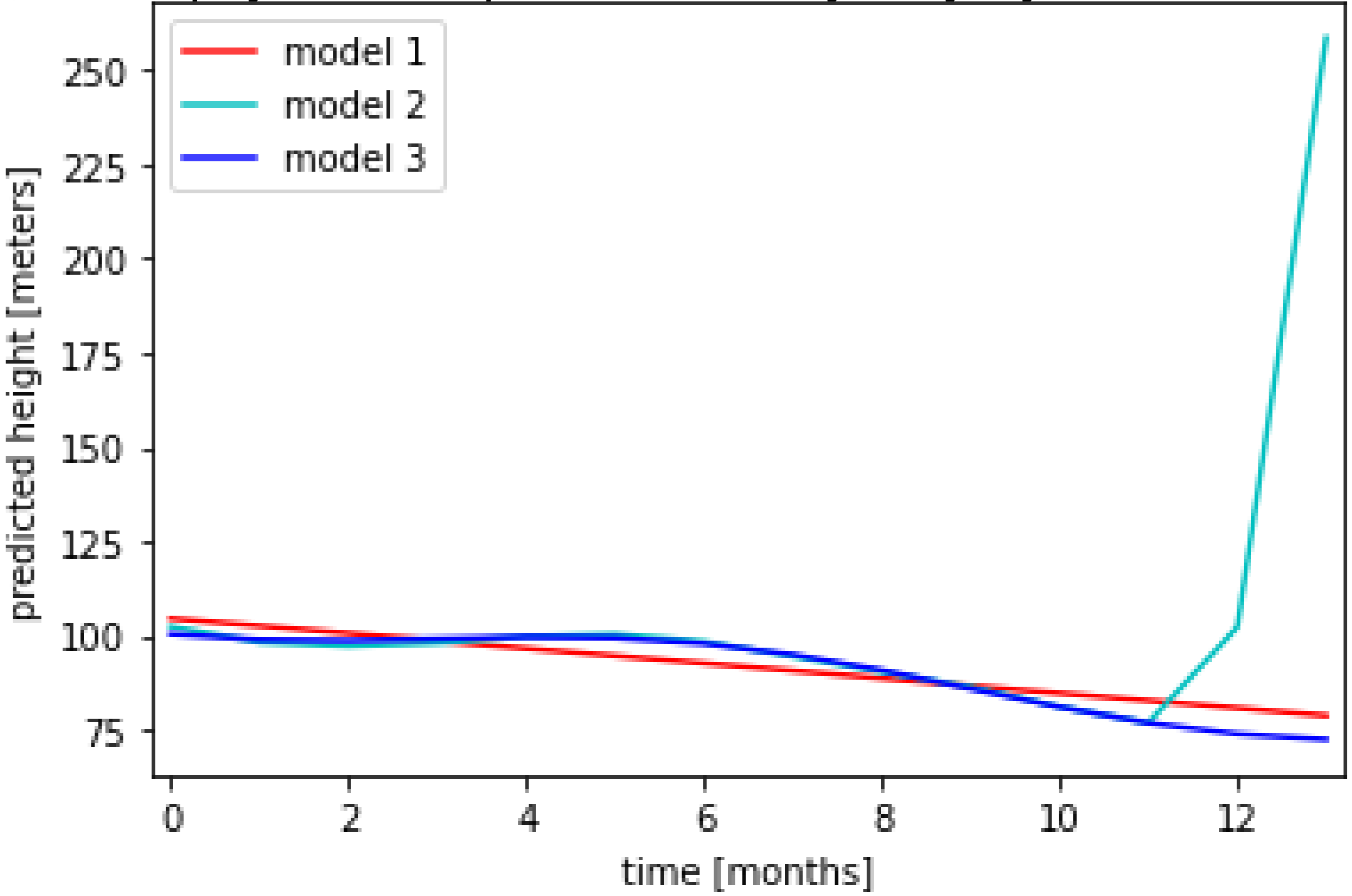
Results from notebooks

- Underfitting: model too simplistic, does not capture the real signal
- Overfitting: nearly perfect fit, but no physical interpretation



Results from notebooks

- Underfitting: model too simplistic, does not capture the real signal
- Overfitting: nearly perfect fit, but no physical interpretation → very risky if you use model for prediction



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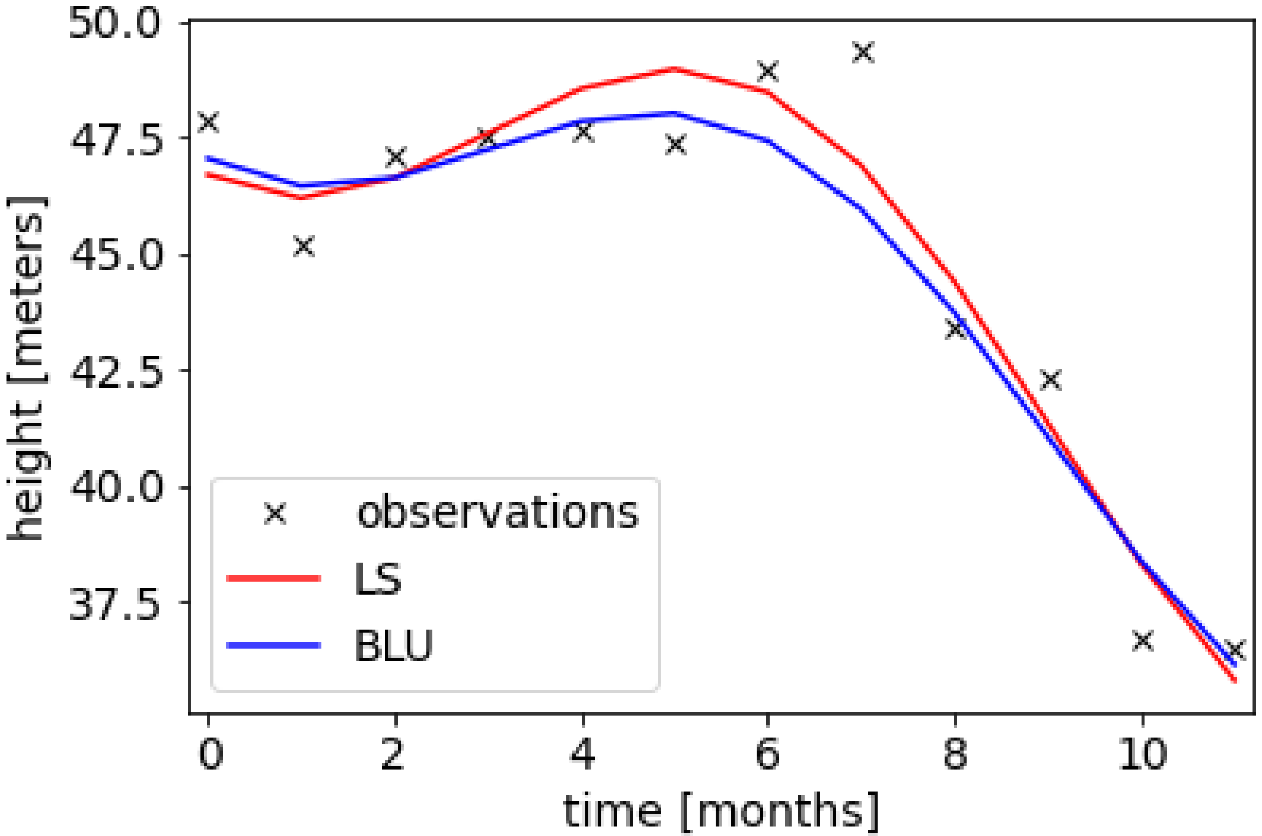
Best linear unbiased estimator = best weighted least squares estimator

Weight matrix is inverse covariance matrix:

Makes sense: high precision → small variance → large weight



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$$Y = \mathrm{A} \cdot \mathrm{x} + \epsilon$$

$$\mathbb{D}(Y) = \Sigma_Y = \Sigma_\epsilon$$

Question that may have popped up:

Where does Σ_Y come from?

→ Calibration:

- Repeated measurements
- Calculate standard deviation

Usually observables are assumed to be independent, since the random errors are independent (error of observation Y_i does not depend on the error of observation Y_j

When would observable be dependent?

- due to signal processing in sensor (often when sampling rate is too high)
- if we use *differential* observables
- if we apply a common correction to our observations which is *stochastic*

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Open questions

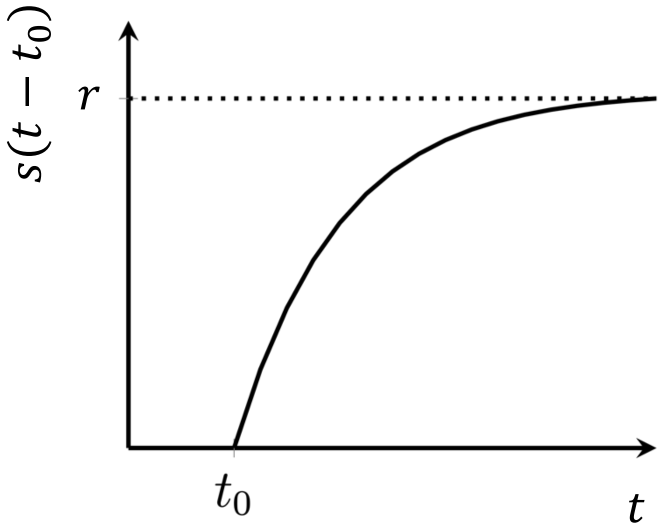
- (How to come up with a model?)
- What if my model is non-linear?
- Does my model really fit?
- Which models fits best?

What if my observation equations are non-linear?

Observed: ground water level rise due to rainfal

$$E(Y_i) = p \cdot \underbrace{r\left(1 - \exp\left(-\frac{t - t_0}{a}\right)\right)}_{s(t-t_0)}$$

- Known parameter:
  - p [m]: constant water inflow during rain event
- Unknown parameters:
  - **scaling parameter a [days] (memory of system),**
  - **response r [m/m] of the aquifer depending on the amount of rainfall**



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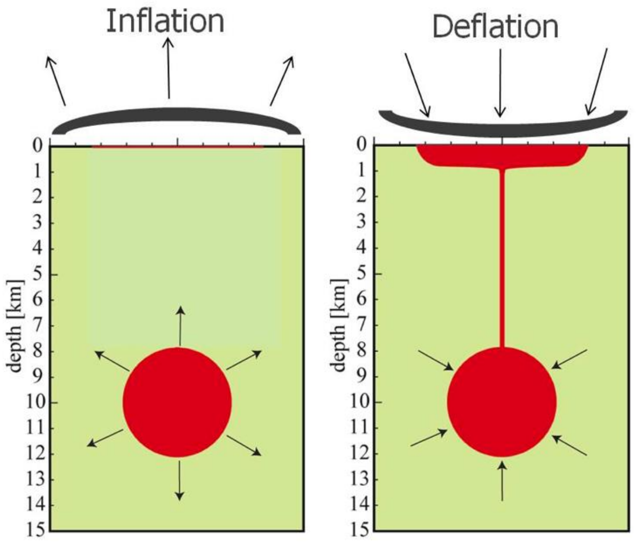
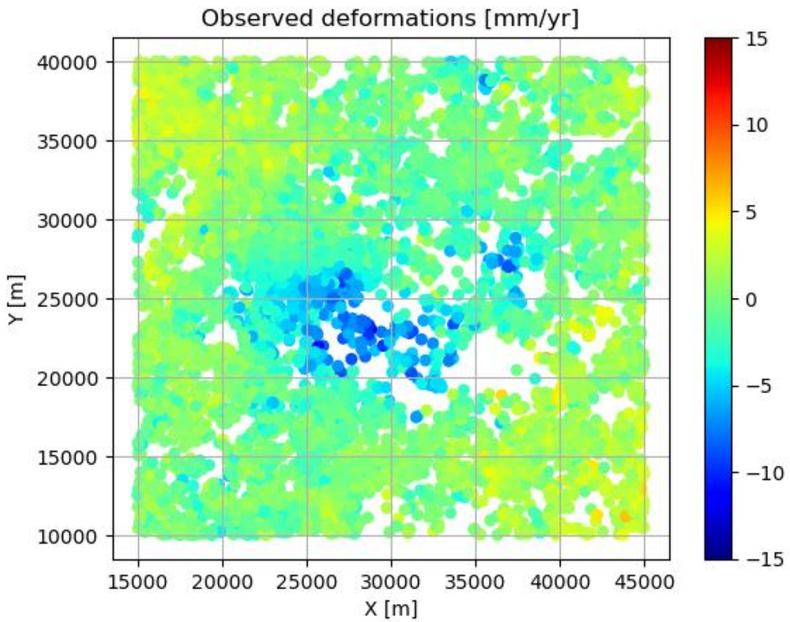
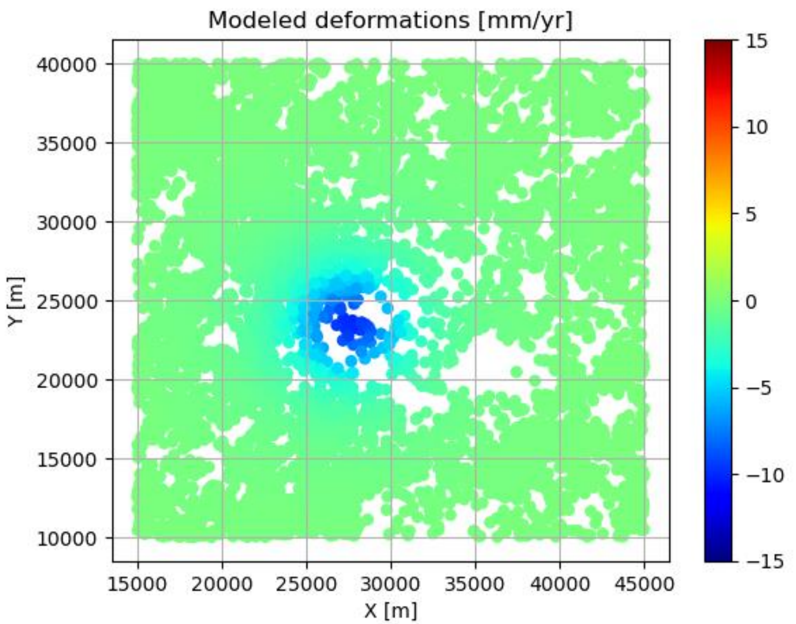
Volcano deformation rates at known locations (x_i, y_i)

$$\mathbb{E}(Y_i) = \frac{0.73\Delta V}{\pi d^2} \cdot \left(1 + \frac{1}{d^2}\left((x_i - x_s)^2 + (y_i - y_s)^2\right)\right)^{-\frac{3}{2}}$$



Unknown parameters: **volume change ΔV, depth of magma chamber d, (x_s, y_s) horizontal coordinates of centre**





Linearized observation equation using 1st order Taylor approximation
1 observation 1 unknown

$$y = q(x){\color{red}+\epsilon} \approx q(x_{[0]}) + \partial_x q(x_{[0]})(x - x_{[0]}){\color{red}+\epsilon}$$

initial guess

for now: omit ϵ from equations

$$\Delta y = y - q(x_{[0]}) \approx \partial_x q(x_{[0]})\underbrace{(x - x_{[0]})}_{\Delta x}$$

observed-minus-computed

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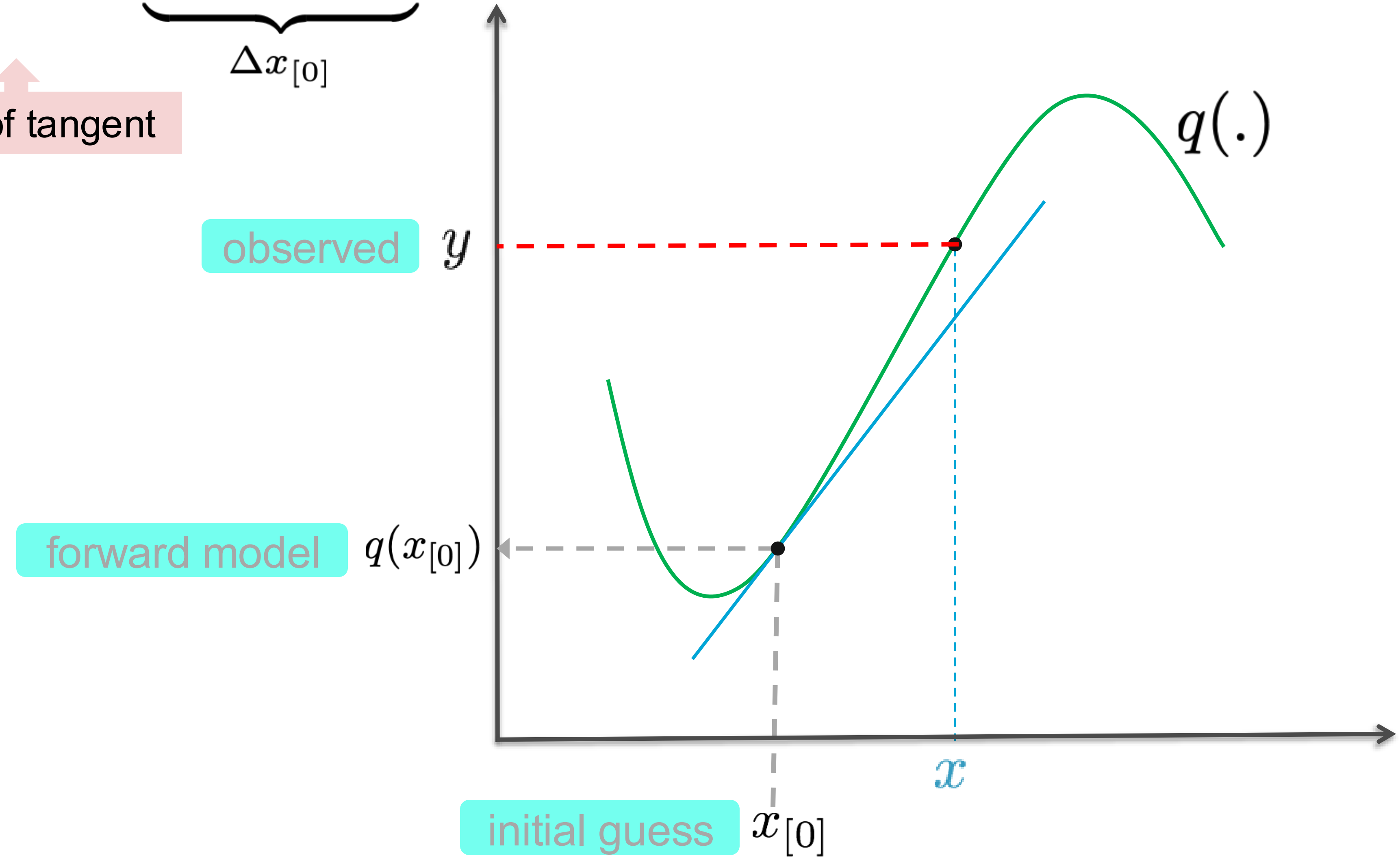
Input:
- observation y
- initial guess $x_{[0]}$

$$\Delta y_{[0]} = y - q(x_{[0]}) \approx \partial_x q(x_{[0]}) \underbrace{(x_{[1]} - x_{[0]})}_{\Delta x_{[0]}}$$

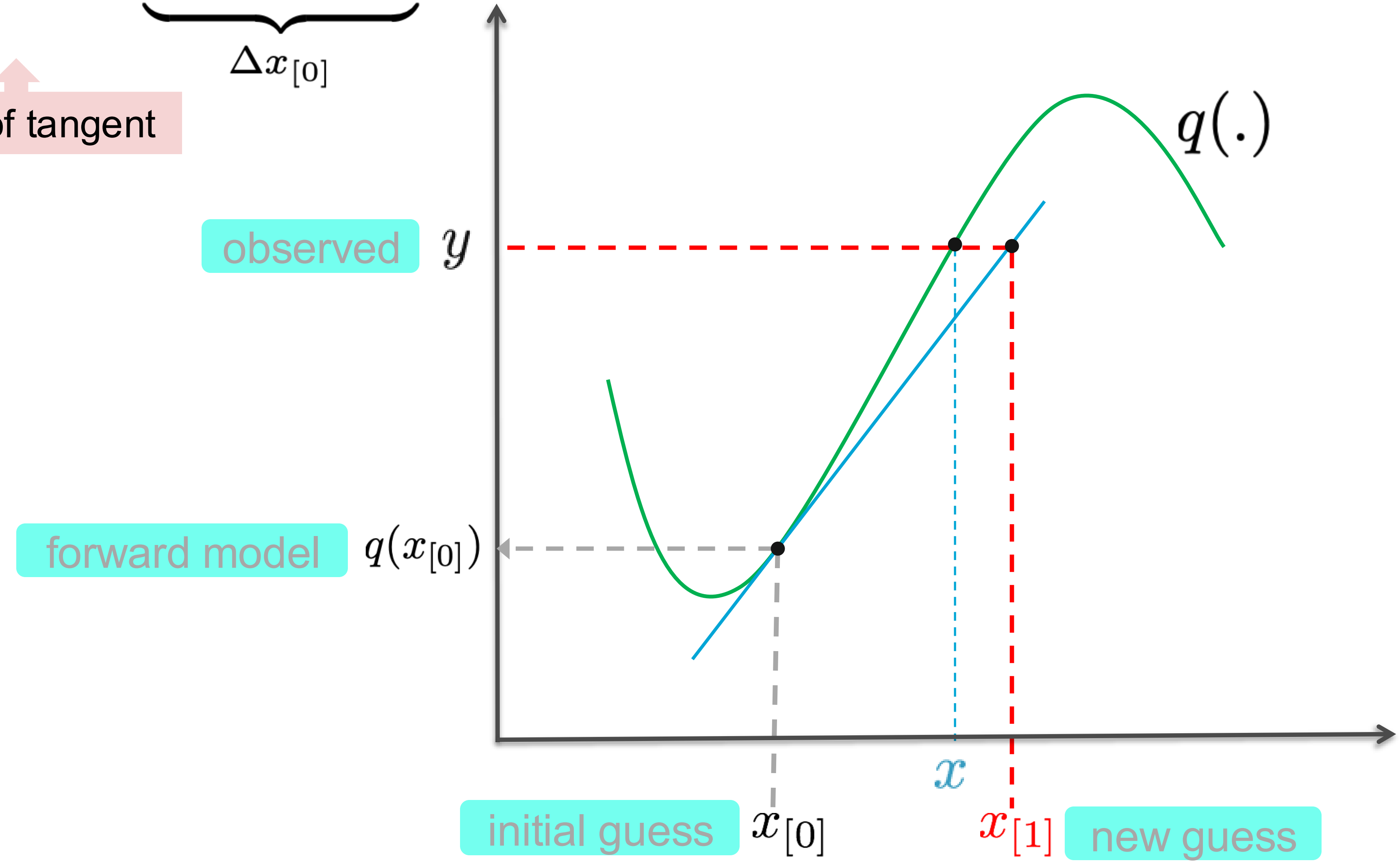
slope of tangent



Input:
- observation y
- initial guess $x_{[0]}$

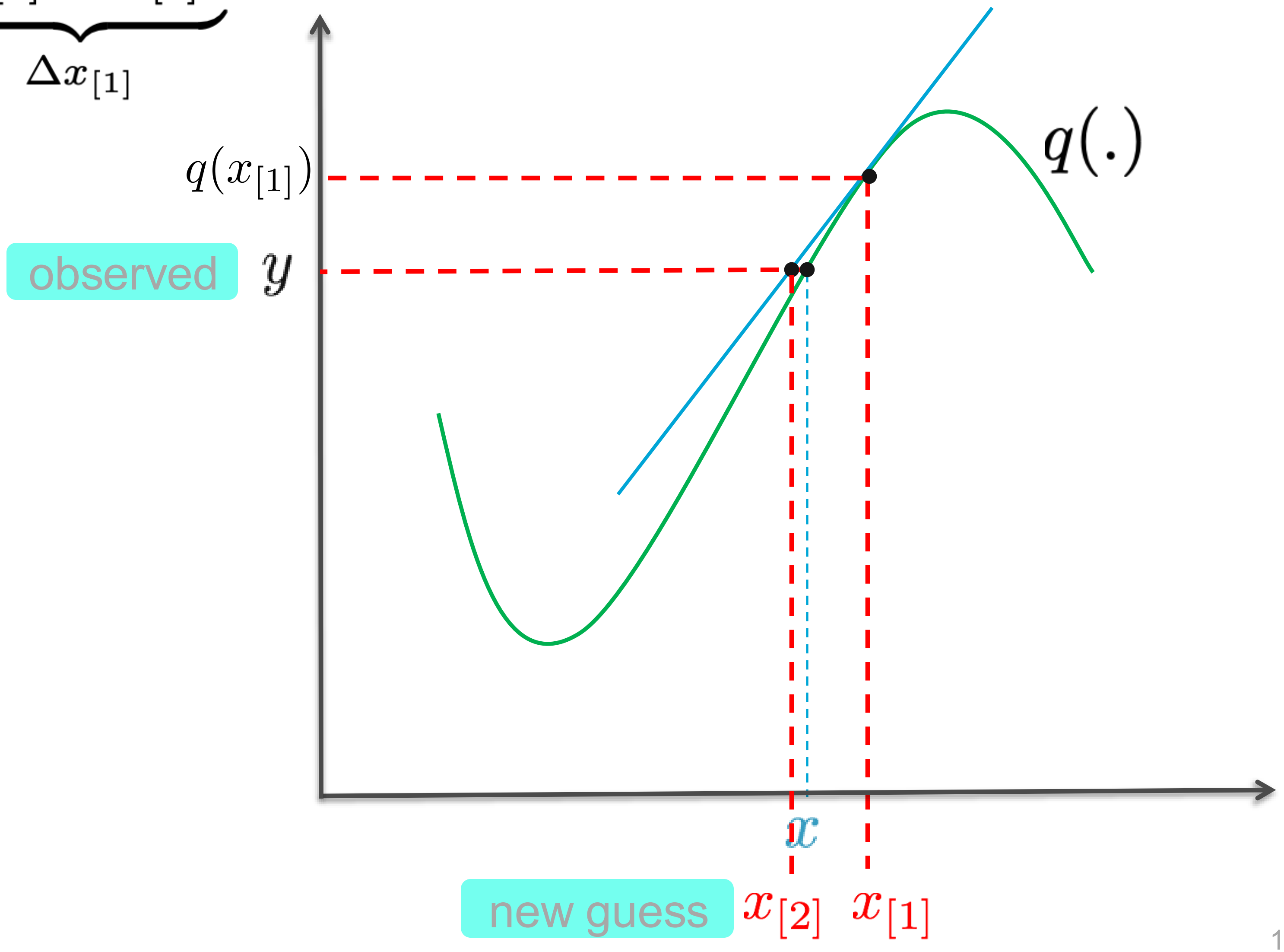
$$\Delta y_{[0]} = y - q(x_{[0]}) \approx \partial_x q(x_{[0]}) \underbrace{(x_{[1]} - x_{[0]})}_{\Delta x_{[0]}}$$

slope of tangent



Input:
- observation y
- new guess $x_{[1]}$

$$\Delta y_{[1]} = y - q(x_{[1]}) \approx \partial_x q(x_{[1]}) \underbrace{(x_{[2]} - x_{[1]})}_{\Delta x_{[1]}}$$



→ Gauss-Newton iteration

Continue until $\Delta x_{[i]}$ is very small

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Linearized observation equation using 1st order Taylor approximation
1 observation n unknowns

$$\Delta y_{[i]} = y - \underset{n \times 1}{q(\mathrm{x}_{[i]})} \approx \partial_{\mathrm{x}} q(\mathrm{x}_{[i]}) \underbrace{(\mathrm{x} - \mathrm{x}_{[i]})}_{\Delta \mathrm{x}_[i]}$$

$$= \begin{bmatrix} \partial_{x_1} q(\mathrm{x}_{[i]}) & \partial_{x_2} q(\mathrm{x}_{[i]}) & \cdots & \partial_{x_n} q(\mathrm{x}_{[i]}) \end{bmatrix} \begin{bmatrix} x_1 - x_{1,[i]} \\ x_2 - x_{2,[i]} \\ \vdots \\ x_n - x_{n,[i]} \end{bmatrix}$$

i is the iteration index

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Non-linear functional model

$$\mathbb{E}\left(\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{bmatrix}\right) = \begin{bmatrix} q_1(\mathrm{x}) \\ q_2(\mathrm{x}) \\ \vdots \\ q_m(\mathrm{x}) \end{bmatrix}$$

Linearized functional model

$$\mathbb{E}\left(\begin{bmatrix} \Delta Y_1 \\ \Delta Y_2 \\ \vdots \\ \Delta Y_m \end{bmatrix}_{[i]}\right) = \underbrace{\begin{bmatrix} \partial_{x_1} q_1(\mathrm{x}_{[i]}) & \partial_{x_2} q_1(\mathrm{x}_{[i]}) & \cdots & \partial_{x_n} q_1(\mathrm{x}_{[i]}) \\ \partial_{x_1} q_2(\mathrm{x}_{[i]}) & \partial_{x_2} q_2(\mathrm{x}_{[i]}) & \cdots & \partial_{x_n} q_2(\mathrm{x}_{[i]}) \\ \vdots & \vdots & & \vdots \\ \partial_{x_1} q_m(\mathrm{x}_{[i]}) & \partial_{x_2} q_m(\mathrm{x}_{[i]}) & \cdots & \partial_{x_n} q_m(\mathrm{x}_{[i]}) \end{bmatrix}} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix}_{[i]}$$

Jacobian ← takes the role of design matrix A

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Gauss-Newton iteration

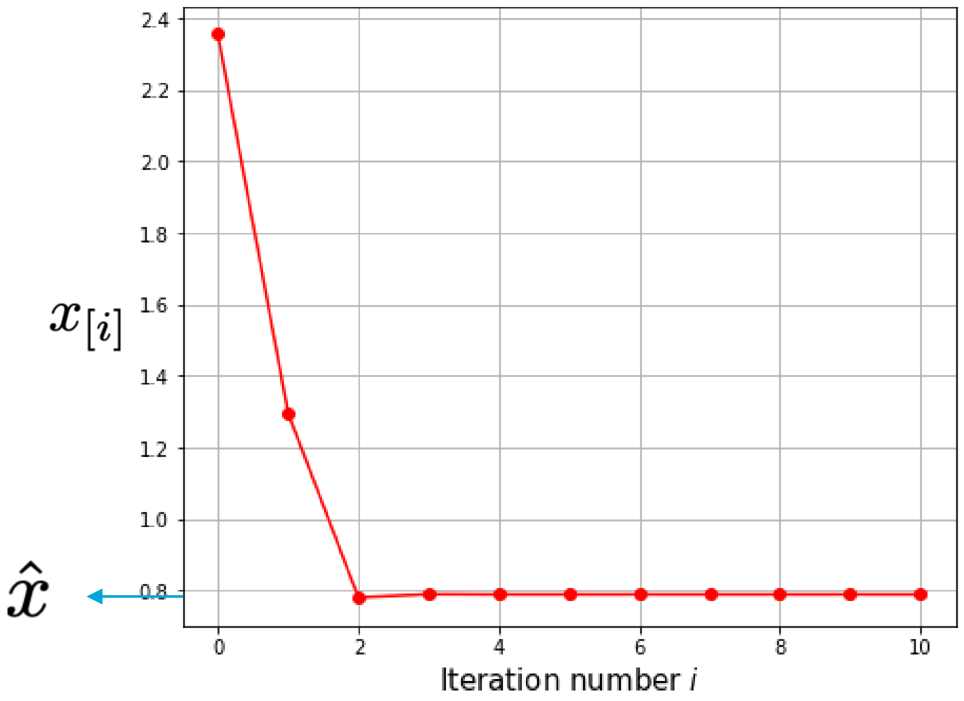
Start with initial guess $\mathrm{x}_{[0]}$, and start iteration with $i = 0$

1. Calculate observed-minus-computed $\Delta \mathrm{y}_{[i]}$
2. Determine the Jacobian
3. Estimate $\Delta \hat{\mathrm{x}}_{[i]}$ by applying BLUE
4. New guess $\mathrm{x}_{[i+1]} = \Delta \hat{\mathrm{x}}_{[i]} + \mathrm{x}_{[i]}$

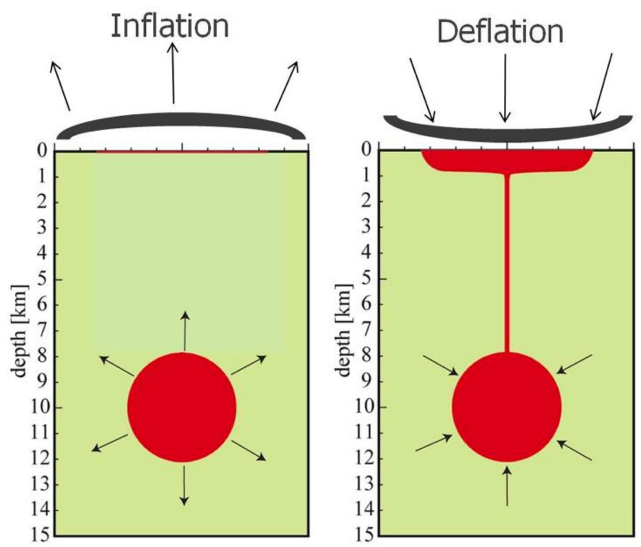
WHEN TO STOP?

$$\mathbb{E}\left(\begin{bmatrix} \Delta Y_1 \\ \Delta Y_2 \\ \vdots \\ \Delta Y_m \end{bmatrix}_{[i]}\right) = \underbrace{\begin{bmatrix} \partial_{x_1} q_1(\mathrm{x}_{[i]}) & \partial_{x_2} q_1(\mathrm{x}_{[i]}) & \cdots & \partial_{x_n} q_1(\mathrm{x}_{[i]}) \\ \partial_{x_1} q_2(\mathrm{x}_{[i]}) & \partial_{x_2} q_2(\mathrm{x}_{[i]}) & \cdots & \partial_{x_n} q_2(\mathrm{x}_{[i]}) \\ \vdots & \vdots & & \vdots \\ \partial_{x_1} q_m(\mathrm{x}_{[i]}) & \partial_{x_2} q_m(\mathrm{x}_{[i]}) & \cdots & \partial_{x_n} q_m(\mathrm{x}_{[i]}) \end{bmatrix}}_{\text{Jacobian} \leftarrow \text{ takes the role of design matrix } \mathrm{A}} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix}_{[i]}$$

Convergence

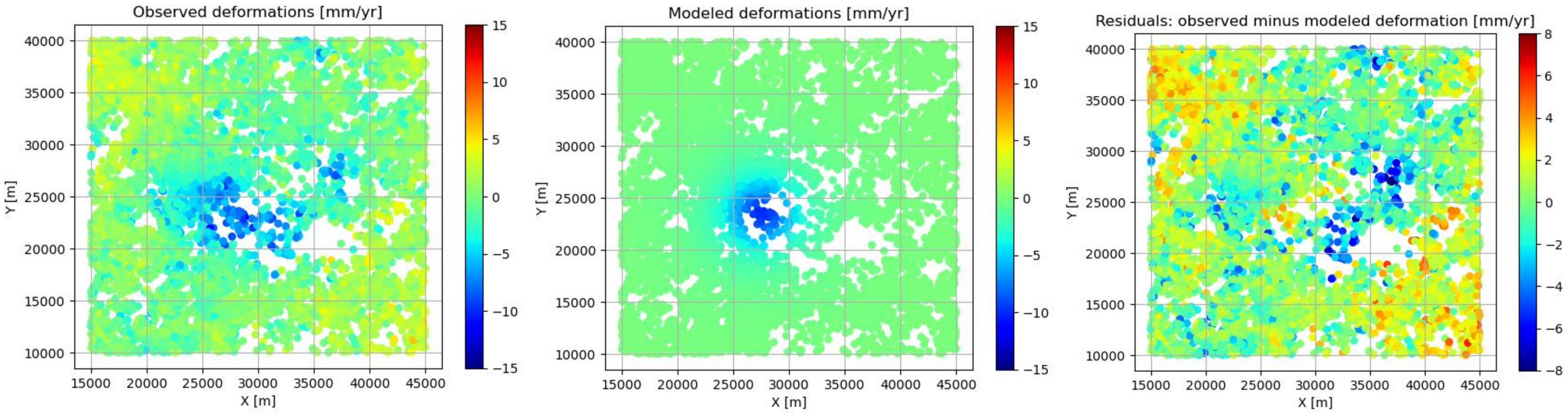


Volcano deformation rates at known locations (x_i, y_i)



$$\mathbb{E}(Y_i) = \frac{0.73\Delta V}{\pi d^2} \cdot \left(1 + \frac{1}{d^2}\left((x_i - x_s)^2 + (y_i - y_s)^2\right)\right)^{-\frac{3}{2}}$$

Unknown parameters: **volume change ΔV, depth of magma chamber d, (x_s, y_s) horizontal coordinates of centre**



Volcano deformation – precision of estimated parameters

observed deformations at (x_i, y_i) as function of volume change, depth, horiz. position of centre

$$\mathbb{E}(Y_i) = \frac{0.73\Delta V}{\pi d^2} \cdot (1 + \frac{1}{d^2}((x_i - x_s)^2 + (y_i - y_s)^2))^{-\frac{3}{2}}$$

$$\begin{bmatrix} \hat{\Delta V} \\ \hat{d} \\ \hat{x}_s \\ \hat{y}_s \end{bmatrix} = \begin{bmatrix} -552352.169\ m^3 \\ 3562.319\ m \\ 27528.535\ m \\ 23540.619\ m \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{\hat{\Delta V}} \\ \sigma_{\hat{d}} \\ \sigma_{\hat{x}_s} \\ \sigma_{\hat{y}_s} \end{bmatrix} = \begin{bmatrix} 1582.769\ m^3 \\ 8.986\ m \\ 8.238\ m \\ 7.239\ m \end{bmatrix}$$

seems large, but look at units, and look at size compared to estimate !

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Gauss-Newton iteration

Stop criterion

$$\Delta\hat{\mathrm{x}}_{[i]}^T \cdot \Sigma_{\hat{\mathrm{x}}}^{-1} \cdot \Delta\hat{\mathrm{x}}_{[i]} < \text{small value}$$

an estimated parameter with small variance should have a relatively small deviation compared to a parameter with large variance

Start with initial guess $\mathrm{x}_{[0]}$, and start iteration with $i = 0$

1. Calculate observed-minus-computed $\Delta\mathrm{y}_{[i]}$
2. Determine the Jacobian
3. Estimate $\Delta\hat{\mathrm{x}}_{[i]}$ by applying BLUE
4. New guess $\mathrm{x}_{[i+1]} = \Delta\hat{\mathrm{x}}_{[i]} + \mathrm{x}_{[i]}$
5. If **stop criterion** is met: set $\hat{\mathrm{x}} = \mathrm{x}_{[i+1]}$ and break, otherwise set $i := i + 1$ and go to step 1

$$\mathbb{E}\left(\begin{bmatrix} \Delta Y_1 \\ \Delta Y_2 \\ \vdots \\ \Delta Y_m \end{bmatrix}_{[i]} \right) = \begin{bmatrix} \partial_{x_1} q_1(\mathrm{x}_{[i]}) & \partial_{x_2} q_1(\mathrm{x}_{[i]}) & \cdots & \partial_{x_n} q_1(\mathrm{x}_{[i]}) \\ \partial_{x_1} q_2(\mathrm{x}_{[i]}) & \partial_{x_2} q_2(\mathrm{x}_{[i]}) & \cdots & \partial_{x_n} q_2(\mathrm{x}_{[i]}) \\ \vdots & \vdots & & \vdots \\ \partial_{x_1} q_m(\mathrm{x}_{[i]}) & \partial_{x_2} q_m(\mathrm{x}_{[i]}) & \cdots & \partial_{x_n} q_m(\mathrm{x}_{[i]}) \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix}_{[i]}$$

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Is it a good fit?

Sensing and observation theory - why

Needed for monitoring and prediction
e.g., natural processes, human-induced deformations, structural health, climate & environment, geo-energy and geo-resources, …

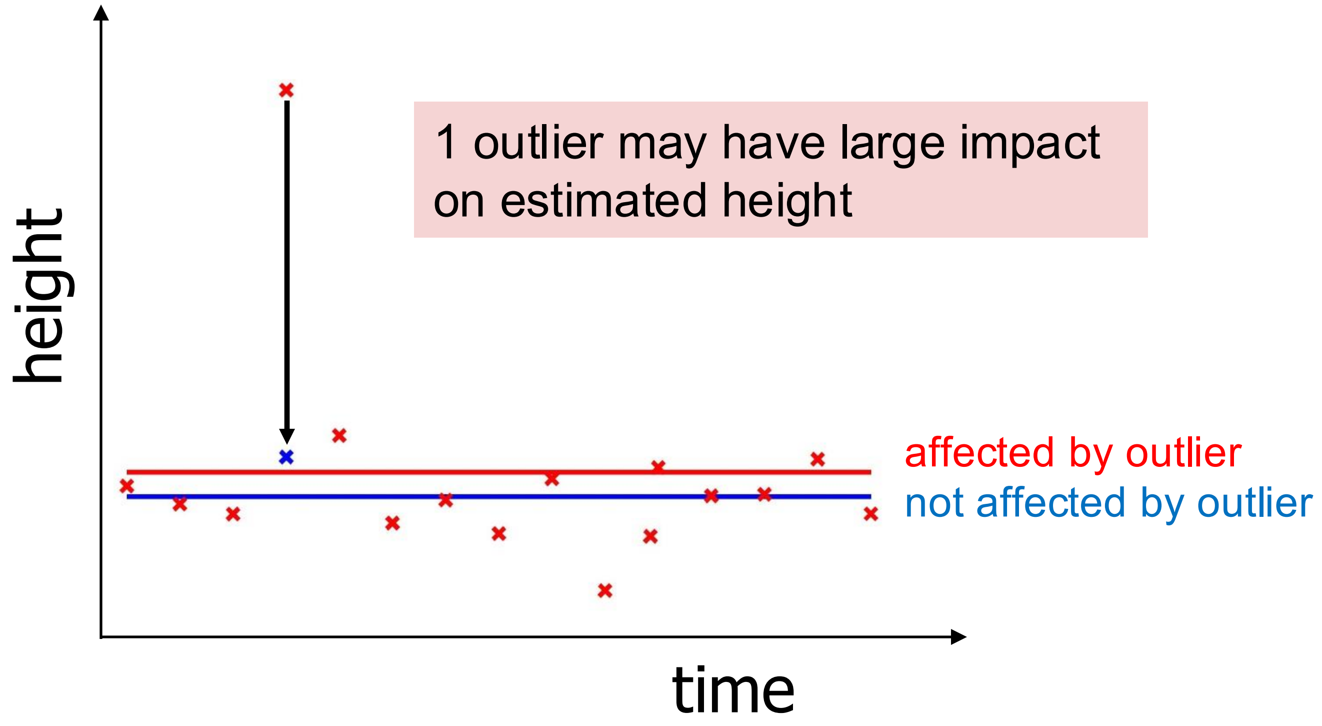
- Process measurements (= observations) to estimate parameters of interest
- In order to use estimation results for further analysis and interpretations (eventually to make decisions)

= uncertainty quantification

= detection of errors in data (outliers, systematic biases)
+ correction / adaption for these errors

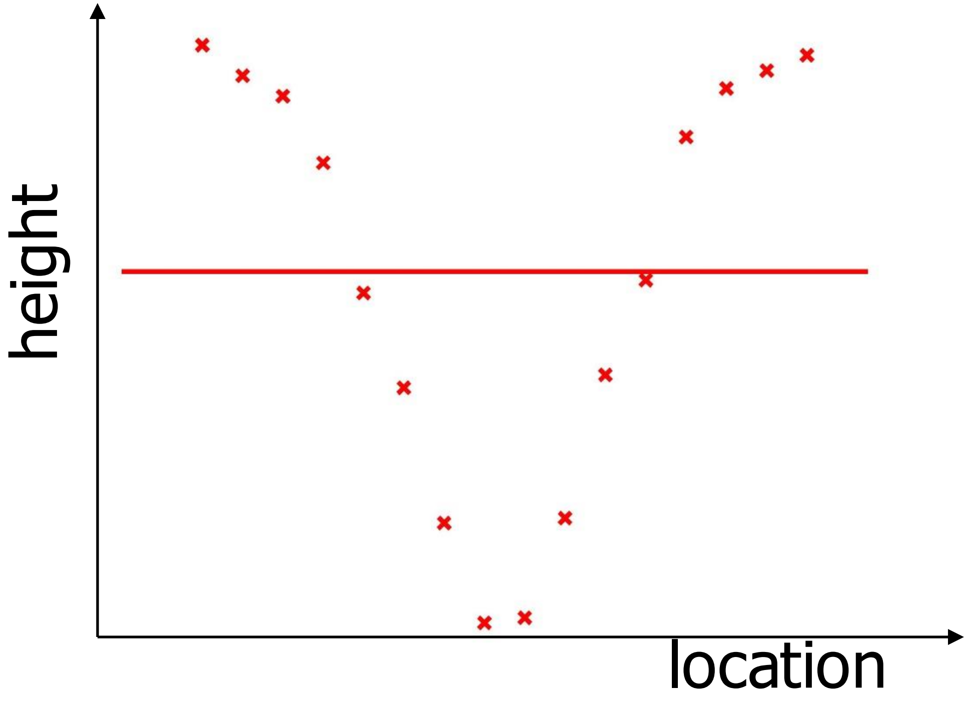
= model validation
- detect model misspecifications
- multiple candidate models → decide which one is best

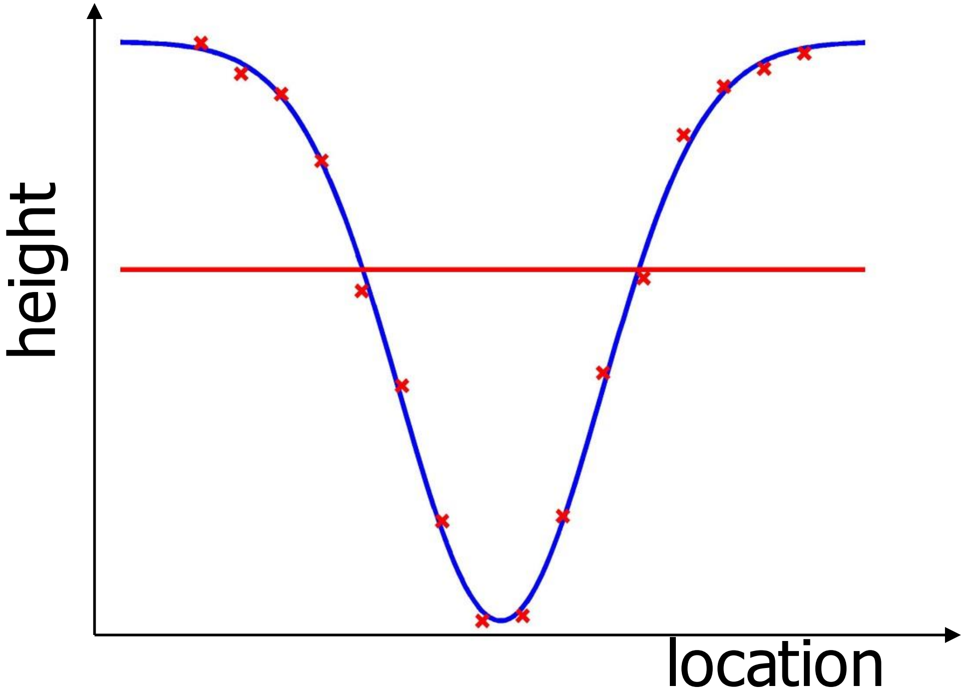
Example: outlier



Example: model misspecification

Wrong model → large residuals
(difference observations and fitted model)





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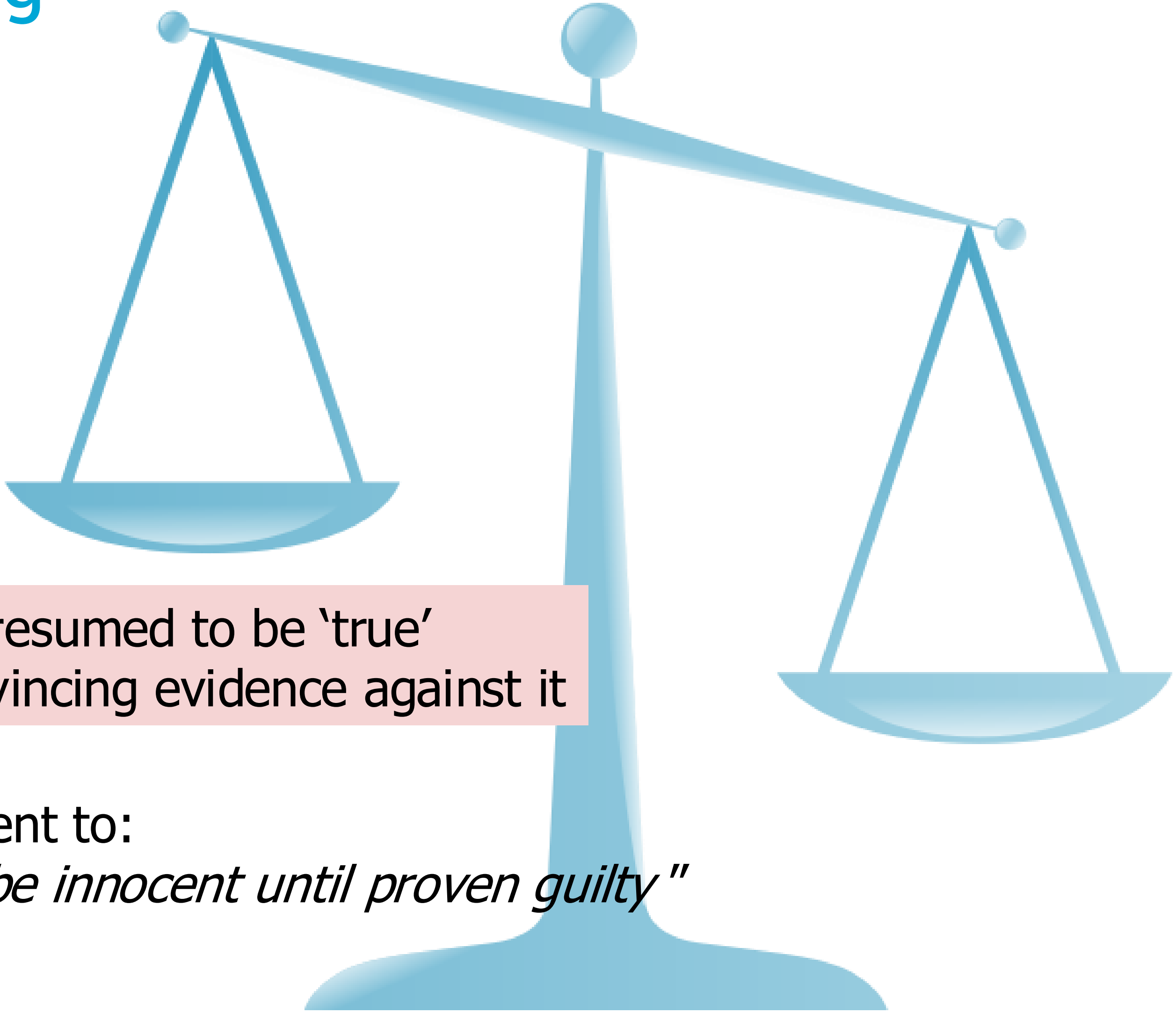
Statistical hypothesis testing

→ test for compliance of model and data

Two competing hypothesis:

- Null hypothesis (nominal model): $\mathcal{H}_0$
- Alternative hypothesis: $\mathcal{H}_a$



Null hypothesis presumed to be 'true'
until data provide convincing evidence against it

equivalent to:
" *the defendant is presumed to be innocent until proven guilty* "

Group assignment

$\mathcal{H}_0$ $$d = d_0 + vt + A\sin(\frac{2\pi t}{365} - \phi),$$

$\mathcal{H}_a$ $$d = d_0 + v\,(1 - \exp\left(\frac{-t}{a}\right)) + A\sin(\frac{2\pi t}{365} - \phi),$$

Apply non-linear least-squares
How to decide between the two models?

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