CEGM1000
Modelling, Uncertainty and Data for Engineers
28-JAN-2025

Formula Sheet

Useful stuff from linear algebra:

Transpose of product	$(\mathrm{A}\cdot\mathrm{B}\cdot\mathrm{C})^T = \mathrm{C}^T\mathrm{B}^T\mathrm{A}^T$
Transpose of symmetric matrix W	$\mathrm{W}^T = \mathrm{W}$
	$\left(\mathrm{A}^T\mathrm{W}\mathrm{A}\right)$ and $\left(\mathrm{A}^T\mathrm{W}\mathrm{A}\right)^{-1}$ are symmetric

Uncertainty propagation laws if $X = q(Y_1, Y_2)$

... for the mean	$\mathbb{E}(X) \approx q(\mu_Y) + \frac{1}{2}\left(\frac{\partial^2 q}{\partial Y_1^2}\right)_0 \sigma_1^2 + \frac{1}{2}\left(\frac{\partial^2 q}{\partial Y_2^2}\right)_0 \sigma_2^2 + \left(\frac{\partial^2 q}{\partial Y_1 \partial Y_2}\right)_0 Cov(Y_1, Y_2)$
... for variance	$\sigma_X^2 \approx \left(\frac{\partial q}{\partial Y_1}\right)_0^2 \sigma_1^2 + \left(\frac{\partial q}{\partial Y_2}\right)_0^2 \sigma_2^2 + 2\left(\frac{\partial q}{\partial Y_1}\right)_0 \left(\frac{\partial q}{\partial Y_2}\right)_0 Cov(Y_1, Y_2)$

Linear propagation laws if $X = \mathrm{L}^T Y$

... for the mean	$\mathbb{E}(X) = \mathrm{L}^T\mathbb{E}(Y)$
... for covariance matrix	$\Sigma_X = \mathrm{L}\Sigma_Y\mathrm{L}^T$

Observables (random)	$Y = \left[Y_1, Y_2, \ldots, Y_m\right]^T$
Observations (realization of Y)	$\mathrm{y} = \left[y_1, y_2, \ldots, y_m\right]^T$
Unknown parameters (deterministic)	$\mathrm{x} = \left[x_1, x_2, \ldots, x_n\right]^T$
Random errors	$\epsilon = \left[\epsilon_1, \epsilon_2, \ldots, \epsilon_m\right]^T$ with $\epsilon \sim N(0, \Sigma_\epsilon)$
Functional model (linear)	$\mathbb{E}(Y) = \mathrm{A}\mathrm{x}$ or $Y = \mathrm{A}\mathrm{x} + \epsilon$
Stochastic model	$\mathbb{D}(Y) = \Sigma_Y = \Sigma_\epsilon$
Estimator of x	$\hat{X}$
Estimate of x (realization of $\hat{X}$)	$\hat{\mathrm{x}}$
Adjusted (or predicted) observations	$\hat{y} = \mathrm{A}\hat{\mathrm{x}}$
Residuals	$\hat{\epsilon} = \mathrm{y} - \mathrm{A}\hat{\mathrm{x}}$
Least squares estimator of x	$\hat{X} = \left(\mathrm{A}^T\mathrm{A}\right)^{-1}\mathrm{A}^T Y$
Weighted least squares estimator of x	$\hat{X} = \left(\mathrm{A}^T\mathrm{W}\mathrm{A}\right)^{-1}\mathrm{A}^T\mathrm{W}Y$
Best linear unbiased estimator (BLUE)	$\hat{X} = \left(\mathrm{A}^T\Sigma_Y^{-1}\mathrm{A}\right)^{-1}\mathrm{A}^T\Sigma_Y^{-1}Y$

Test statistics (with distribution under $\mathcal{H}_0$):

Generalized likelihood ratio test statistic	$T_q = \hat{\epsilon}^T\Sigma_Y^{-1}\hat{\epsilon} - \hat{\epsilon}_a^T\Sigma_Y^{-1}\hat{\epsilon}_a \sim \chi^2(q, 0)$
Overall model test statistic	$T_{q=m-n} = \hat{\epsilon}^T\Sigma_Y^{-1}\hat{\epsilon} \sim \chi^2(q, 0)$
w-test	$W = \dfrac{\mathrm{C}^T\Sigma_Y^{-1}\hat{\epsilon}}{\sqrt{\mathrm{C}^T\Sigma_Y^{-1}\Sigma_{\hat{\epsilon}}\Sigma_Y^{-1}\mathrm{C}}} \sim N(0,1)$

Taylor Series: $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \ldots + \frac{h^n}{n!}f^n(x) + O(h^{n+1})$

Backward Euler: $f'(x) = \dfrac{f(x) - f(x-h)}{h} + O(h)$

Forward Euler: $f'(x) = \dfrac{f(x+h) - f(x)}{h} + O(h)$

Distributions

Probability density functions, cumulative distribution functions and parameters of some relevant distributions.

Distribution	**PDF**	**CDF**	**Mean and variance**
Exponential	$f_X(x) = \lambda e^{-\lambda x}$	$F_X(x) = 1 - e^{-\lambda x}$	$\mu = \frac{1}{\lambda}$ $\sigma^2 = \frac{1}{\lambda^2}$
Gumbel	$f_X(x) = \frac{1}{\beta} e^{-\left(z + e^{-z}\right)}$, where $z = \frac{x-\alpha}{\beta}$	$F_X(x) = e^{-e^{-z}}$	$\mu = \alpha + \beta\gamma,\ \gamma = 0.5772$ $\sigma^2 = \frac{\pi^2}{6}\beta^2$
Normal	$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}$	$F_X(x) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)$	$\mu = \mu$ $\sigma^2 = \sigma^2$
Uniform	$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$	$F_X(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } x \in [a,b] \\ 1 & \text{for } x > b \end{cases}$	$\mu = \frac{1}{2}(a+b)$ $\sigma^2 = \frac{1}{12}(b-a)^2$

Other relevant equations:

The empirical or sample mean based on m outcomes (or: realizations) x_i can be computed as:

$$\hat{\mu}_X = \frac{1}{m}\sum_{i=1}^{m} x_i$$

The unbiased sample variance can be computed as:

$$\hat{\sigma}_X^2 = \frac{1}{m-1}\sum_{i=1}^{m}(x_i - \hat{\mu}_X)^2$$

The covariance is given by

$$Cov(X_1, X_2) = \mathbb{E}([X_1 - \mathbb{E}(X_1)][X_2 - \mathbb{E}(X_2)])$$

The correlation coefficient is given by

$$\rho(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}}$$

Table for standard normal distribution $Z \sim N(0,1)$

Recall that if $X \sim N(\mu, \sigma^2)$ then $Z = \dfrac{X-\mu}{\sigma} \sim N(0,1)$

The table shows one-sided (**right-hand**) probabilities α as function of the critical value k_α, i.e. $\alpha = P(Z \geq k_\alpha)$.
Values of k_α: value up to first decimal in first column, second decimal in first row. Example: $\alpha = 0.0250$ for $k_\alpha = 1.96$.

k_α	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002

Table for central χ^2 distribution

Central χ^2 distribution: critical value k_α as function of level of significance α and degrees of freedom q.
Example: $\alpha = 0.010$ and $q = 10$ yield $k_\alpha = 23.2093$.

α	0.9990	0.9950	0.9900	0.9750	0.9500	0.9000	0.1000	0.0500	0.0250	0.0100	0.0050	0.0010
q												
1	0.0000	0.0000	0.0002	0.0010	0.0039	0.0158	2.7055	3.8415	5.0239	6.6349	7.8794	10.8276
2	0.0020	0.0100	0.0201	0.0506	0.1026	0.2107	4.6052	5.9915	7.3778	9.2103	10.5966	13.8155
3	0.0243	0.0717	0.1148	0.2158	0.3518	0.5844	6.2514	7.8147	9.3484	11.3449	12.8382	16.2662
4	0.0908	0.2070	0.2971	0.4844	0.7107	1.0636	7.7794	9.4877	11.1433	13.2767	14.8603	18.4668
5	0.2102	0.4117	0.5543	0.8312	1.1455	1.6103	9.2364	11.0705	12.8325	15.0863	16.7496	20.5150
6	0.3811	0.6757	0.8721	1.2373	1.6354	2.2041	10.6446	12.5916	14.4494	16.8119	18.5476	22.4577
7	0.5985	0.9893	1.2390	1.6899	2.1673	2.8331	12.0170	14.0671	16.0128	18.4753	20.2777	24.3219
8	0.8571	1.3444	1.6465	2.1797	2.7326	3.4895	13.3616	15.5073	17.5345	20.0902	21.9550	26.1245
9	1.1519	1.7349	2.0879	2.7004	3.3251	4.1682	14.6837	16.9190	19.0228	21.6660	23.5894	27.8772
10	1.4787	2.1559	2.5582	3.2470	3.9403	4.8652	15.9872	18.3070	20.4832	23.2093	25.1882	29.5883
11	1.8339	2.6032	3.0535	3.8157	4.5748	5.5778	17.2750	19.6751	21.9200	24.7250	26.7568	31.2641
12	2.2142	3.0738	3.5706	4.4038	5.2260	6.3038	18.5493	21.0261	23.3367	26.2170	28.2995	32.9095
13	2.6172	3.5650	4.1069	5.0088	5.8919	7.0415	19.8119	22.3620	24.7356	27.6882	29.8195	34.5282
14	3.0407	4.0747	4.6604	5.6287	6.5706	7.7895	21.0641	23.6848	26.1189	29.1412	31.3193	36.1233
15	3.4827	4.6009	5.2293	6.2621	7.2609	8.5468	22.3071	24.9958	27.4884	30.5779	32.8013	37.6973
16	3.9416	5.1422	5.8122	6.9077	7.9616	9.3122	23.5418	26.2962	28.8454	31.9999	34.2672	39.2524
17	4.4161	5.6972	6.4078	7.5642	8.6718	10.0852	24.7690	27.5871	30.1910	33.4087	35.7185	40.7902
18	4.9048	6.2648	7.0149	8.2307	9.3905	10.8649	25.9894	28.8693	31.5264	34.8053	37.1565	42.3124
19	5.4068	6.8440	7.6327	8.9065	10.1170	11.6509	27.2036	30.1435	32.8523	36.1909	38.5823	43.8202
20	5.9210	7.4338	8.2604	9.5908	10.8508	12.4426	28.4120	31.4104	34.1696	37.5662	39.9968	45.3147
21	6.4467	8.0337	8.8972	10.2829	11.5913	13.2396	29.6151	32.6706	35.4789	38.9322	41.4011	46.7970
22	6.9830	8.6427	9.5425	10.9823	12.3380	14.0415	30.8133	33.9244	36.7807	40.2894	42.7957	48.2679
23	7.5292	9.2604	10.1957	11.6886	13.0905	14.8480	32.0069	35.1725	38.0756	41.6384	44.1813	49.7282
24	8.0849	9.8862	10.8564	12.4012	13.8484	15.6587	33.1962	36.4150	39.3641	42.9798	45.5585	51.1786
25	8.6493	10.5197	11.5240	13.1197	14.6114	16.4734	34.3816	37.6525	40.6465	44.3141	46.9279	52.6197
26	9.2221	11.1602	12.1981	13.8439	15.3792	17.2919	35.5632	38.8851	41.9232	45.6417	48.2899	54.0520
27	9.8028	11.8076	12.8785	14.5734	16.1514	18.1139	36.7412	40.1133	43.1945	46.9629	49.6449	55.4760
28	10.3909	12.4613	13.5647	15.3079	16.9279	18.9392	37.9159	41.3371	44.4608	48.2782	50.9934	56.8923
29	10.9861	13.1211	14.2565	16.0471	17.7084	19.7677	39.0875	42.5570	45.7223	49.5879	52.3356	58.3012
30	11.5880	13.7867	14.9535	16.7908	18.4927	20.5992	40.2560	43.7730	46.9792	50.8922	53.6720	59.7031
40	17.9164	20.7065	22.1643	24.4330	26.5093	29.0505	51.8051	55.7585	59.3417	63.6907	66.7660	73.4020
50	24.6739	27.9907	29.7067	32.3574	34.7643	37.6886	63.1671	67.5048	71.4202	76.1539	79.4900	86.6608
60	31.7383	35.5345	37.4849	40.4817	43.1880	46.4589	74.3970	79.0819	83.2977	88.3794	91.9517	99.6072
70	39.0364	43.2752	45.4417	48.7576	51.7393	55.3289	85.5270	90.5312	95.0232	100.4252	104.2149	112.3169
80	46.5199	51.1719	53.5401	57.1532	60.3915	64.2778	96.5782	101.8795	106.6286	112.3288	116.3211	124.8392
90	54.1552	59.1963	61.7541	65.6466	69.1260	73.2911	107.5650	113.1453	118.1359	124.1163	128.2989	137.2084
100	61.9179	67.3276	70.0649	74.2219	77.9295	82.3581	118.4980	124.3421	129.5612	135.8067	140.1695	149.4493