Exam Q1 R

CEGM1000 Modelling, Uncertainty and Data for Engineers EXAM 24/25 · 4 exercises · 100.0 points

Propagation of Uncertainty #42674783

20 pts · Last updated 15 Feb, 16:35

A speed detection system is used to monitor the speed of vehicles in a traffic flow. The speed of the vehicle can then be determined by analyzing the Doppler shift in the returned signal, which depends on the propagation speed c of the signal in the air. This propagation speed varies with air temperature T and humidity H as:

$$c = 331.3 + 0.6T + 0.0124H^2$$

where c is in m/s, T is in °C, and H is in g/m³.

Both temperature and humidity are random variables due to environmental fluctuations.
We are interested in the precision of the propagation speed, where it is known that:
$\mu T = 25$°C, $\sigma_T = 3$ °C
$\mu_H = 10$ g/m³, $\sigma_H = 2$ g/m³.
$Cov\,(T, H) = 0$

Text

Approximate the standard deviation of the propagation speed (give your answer to 2 decimal places). Show how you arrived at your answer.

8 pts · Open · 1 3/5 Page

Correct terms: $\frac{dc}{dT} = 0.6$ and $\frac{dc}{dH} = 0.0248H$	2 pts
Correct implementation of propagation: $\sigma_c = 0.6^2 \cdot \sigma_T^2 + (0.0248 \cdot \mu_H)^2\, \sigma_H^2 + 0 = 3.49$	4 pts
Final solution: $\sigma_c = \sqrt{3.49} = 1.87$	2 pts
Mistake in terms	-1 pts
mistake in implementation	-2 pts
Mistake in final solution	-1 pts

Would it have more impact to reduce σ_H or σ_T , in order to reduce σ_C,? Explain your answer.

4 pts · Open · 3/10 Page

Correct selection of Sigma_T	2 pts
Explanation of propagation law being used with reference to the squared values 0.6^2 and 0.248^2	2 pts

You find that for the humidity sensor, only 5 measurements were taken during the day. In contrast, the temperature sensor measured the temperature every hour.

Please answer the following questions, you only need to write 1-2 sentences for each:

- Which source of uncertainty affects both of them?
- Which source of uncertainty affects the humidity sensor to a larger degree than the temperature sensor?
- Describe how and if these sources of uncertainty can be reduced.

5 pts · Open · 1/2 Page

Both: Aleatoric, explanation is also ok if there is no name	1 pt
Humidity sensor: Epistemic, explanation is also ok if there is no name	1 pt
Aleatoric cannot be mitigated as it is inherent in nature, but model can be modified better represent the natural phenomenon	1.5 pts
Epistemic: Gather more data, improve data gathering process or get better equipment	1.5 pts
weak answer for bullet point 3	-0.5 pts

Which of the following is **not true** about propagation of uncertainty?

3 pts · Multiple choice · 3 alternatives

- ○ Independent variables always lead to additive variances.
- ○ Covariance terms are zero when variables are uncorrelated.
- ◉ **The propagation of uncertainty is only applicable to linear functions.**

  Feedback

  This is false. Uncertainty propagation can apply to non-linear functions as well. In fact, there are well-established methods (e.g., using Taylor series expansions) to propagate uncertainties through non-linear functions.

Feedback

Feedback when the question is answered correctly

Feedback when the question is answered partially correctly

Feedback when the question is answered incorrectly

Observation Theory #42674725

27 pts · Last updated 31 Oct, 00:27

A scientist is studying the decay of a radioactive isotope in groundwater to estimate its half-life. The concentration of the isotope $C(t)$ at time t follows the exponential decay law:

$$C(t) = C_0 e^{-\lambda t}$$

where:

- C_0 is the initial concentration of the isotope at $t = 0$,
- λ is the decay constant,
- t is the time since the sample was collected.
- The scientist measures the concentrations C_i at 6 different times, creating 6 observations with standard deviation σ_c. To simplify the analysis, the geoscientist linearizes the equation by taking the natural logarithm:

$$\ln(C_i) = \ln(C_0) - \lambda t_i.$$

Where the following is defined:

$$Y_i = \ln(C_i), \quad a = \ln(C_0), \quad b = -\lambda.$$

Text

Give the functional model in the form $\mathbb{E}(\mathbf{Y}) = A\mathbf{x}$ and specifying the following:

- Provide the sizes and shapes of the A matrix and vectors $\mathbf{Y}$ and $\mathbf{x}$.
- The redundancy of the system.

6 pts · Open · 1/2 Page

Model answer

$$E\begin{bmatrix} Y_1 \\ \vdots \\ Y_6 \end{bmatrix} = \begin{bmatrix} t_1 & 1 \\ \vdots & \vdots \\ t_6 & 1 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix}$$

The A matrix has 6 rows and 2 columns.
The Y matrix has 6 rows and 1 column.
The x matrix has 2 rows and 1 column.

The redundancy of the system is q=m-n=6-2=4

correct A and shape	2 pts
correct x and size	1 pt
redundancy 6-2=4	2 pts
correct y and size	1 pt

What happens to a model and errors when the redundancy $q = 0$?
Multiple answers possible.

4 pts · Multiple choice · 5 alternatives

- [ ] The model has extra flexibility, which allows for better error estimation
- [ ] The system cannot be solved, because there is no solution
- [ ] The model becomes statistically reliable for prediction, eliminating the impact of noise
- [x] **The model perfectly fits the data, and the residuals become 0**
- [x] **The model uses all available data, but cannot validate the parameters due to a lack of redundancy.**

Feedback

Feedback when the question is answered correctly

Feedback when the question is answered partially correctly

Feedback when the question is answered incorrectly

Assume that measurement 3 contains a bias of size ∇, making it an outlier. Formulate the alternative hypothesis by modifying the functional model to account for this bias.

4 pts · Open · 3/5 Page

Model answer

$$E\begin{bmatrix} Y_1 \\ \vdots \\ Y_6 \end{bmatrix} = \begin{bmatrix} t_1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ t_3 & 1 & 1 \\ \vdots & \vdots & \vdots \\ t_6 & 1 & 0 \end{bmatrix} \begin{bmatrix} b \\ a \\ \nabla \end{bmatrix}$$

correctly expanding A and x to include bias — 4 pts

For the propagation of uncertainty, we define the non-linear variance propagation law for $X = q\,(Y)$ as:

$$\sigma_X^2 = \left(\frac{\partial q}{\partial Y}\right)_0^2 \sigma_Y^2$$

Considering the transformed observations:

$$Y_i = \ln(C_i)$$

Derive the stochastic model.

4 pts · Open · 7/10 Page

Model answer

The linearised function:

$Y = \ln(C_i), \quad X = q(Y) = q(\ln(C_i))$

The variance propagation law is:

$$\sigma_X^2 = \left(\frac{\partial q}{\partial Y}\right)^2 \sigma_Y^2$$

Therefore:

$$\sigma_X^2 = \sigma_c^2 \frac{1}{C_i^2}$$

The stochastic model becomes:

$$D(Y) = \sigma_c^2 \begin{bmatrix} \frac{1}{C_1^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{C_2^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C_3^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{C_4^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{C_5^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{C_6^2} \end{bmatrix}$$

correct partial derivative	2 pts
correct application of prop law	2 pts
small mistake	-1 pts

To decide between the null hypothesis H_0 and the alternative hypothesis H_a, the scientist conducts a hypothesis test. The test statistic T_q for the hypothesis test has already been computed as:

$$T_q = 6.81$$

Answer the following questions:

- What type of hypothesis test should the scientist use in this scenario?
- Using a false alarm probability of 5%, compute and demonstrate if the Null hypothesis is accepted or rejected

6 pts · Open · New page · 3/5 Page

GLRT, comparing two hypotheses	2 pts
q=3-2=1 (difference between variables in H_0 and H_a)	1 pt
$k_a = 3.84$, from X^2 table	1 pt
reject H_0 as $T_q > K_a$	2 pts

What does **Best Linear Unbiased Estimator (BLUE)** aim to minimize in a linear regression model?

3 pts · Multiple choice · 4 alternatives

- ○ The sum of squared residuals.
- ◉ **The variance of the estimator.**
- ○ The observation noise.
- ○ The sum of absolute residuals.

Feedback

Feedback when the question is answered correctly

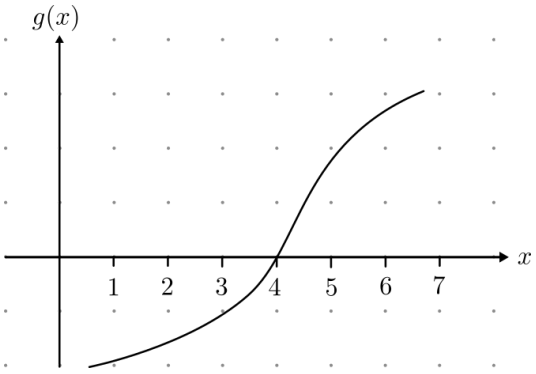
Feedback when the question is answered partially correctly

Feedback when the question is answered incorrectly

Numerical Modelling #41774685

23 pts · Last updated 10 Feb, 19:33

Below you can see a graphical representation for a given function $g\left(x\right)$. We use the Newton-Raphson method to approximate the root of the function $g\left(x\right)=0$



Text

Assuming an initial estimation of $x=5$, indicate the second estimation of the value of x. Provide a sketch in the answer box below (and the next page) to demonstrate your estimated value of x.

Note: you don't need to make explicit calculations, as you can solve this graphically. Your sketch should clearly indicate how you arrived at your estimated value of x. Markings on the figure above will not be graded.

6 pts · Open · 1 2/5 Page

Drawing the tangent line at $x_0=5$, ensuring it properly touches the curve and extends to the x-axis	3 pts
finding the intersection with the x-axis and estimating $x_1\approx 3.5$	3 pts
roughly estimates but its noticeably off (e.g. 3 or below or 4 and above)	2 pts
sloppy graph without clear intention	-1 pts
tangent line incorrectly placed	-1 pts
wrong estimation of x_0 or x_1	-1 pts

Derive the Backward Difference expression **second order accurate** of the **first derivative** around the point x_i. Assume that the distance between points is equidistant.

8 pts · Open · 9/10 Page

We start with the Taylor series expansion of $f(x)$ around x_i:

$$f(x) = f(x_i) + (x - x_i)f'(x_i) + \frac{(x - x_i)^2}{2!} f''(x_i) + \frac{(x - x_i)^3}{3!} f'''(x_i) + \cdots$$

Expanding for $f(x_{i-1})$, gives equation (1):

$$f(x_{i-1}) = f(x_i) - \Delta x f'(x_i) + \frac{\Delta x^2}{2} f''(x_i) - \frac{\Delta x^3}{3!} f'''(x_i) + \dots$$

Expanding for $f(x_{i-2})$, gives equation (2):

$$f(x_{i-2}) = f(x_i) - 2\Delta x f'(x_i) + \frac{4\Delta x^2}{2} f''(x_i) - \frac{8\Delta x^3}{3!} f'''(x_i) + \dots$$

Now, subtract Equation (2) - 4 × Equation (1):

$$-4f(x_{i-1}) + f(x_{i-2}) = -3f(x_i) + 2\Delta x f'(x_i) + O(\Delta x^3)$$

Solving for $f'(x_i)$:

$$f'(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2\Delta x} + O(\Delta x^2)$$

8 pts

Correctly expands series expansion for both $f\left(x_{\{i-1\}}\right), f\left(x_{\{i-2\}}\right)$	3 pts
Correctly sets up Backwards difference. Recognises the need to subtract the equations $-4f(x_{i-1}) + f(x_{i-2})$	3 pts
Correct Final Backward Difference Formula $\frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2\Delta x} + O(\Delta x^2)$	2 pts
Minor algebraic mistakes, but the overall structure is correct.	-1 pts
missing key terms	-1 pts
small error	-1 pts

Select the correct classification of the following equation:

$$\frac{d^3y}{dx^3} - x\frac{d^2y}{dx^2} + y = 0$$

3 pts · Multiple choice · 8 alternatives

- ○ First-order and linear ODE
- ◉ **Third-order and linear ODE**
- ○ First-order and non-linear ODE
- ○ Third-order and non-linear ODE
- ○ First-order linear PDE
- ○ Third- order linear PDE
- ○ First- order and non-linear PDE
- ○ Third- order and non-linear PDE

Feedback

Feedback when the question is answered correctly

Feedback when the question is answered partially correctly

Feedback when the question is answered incorrectly

Multi stage methods ... (select all that correctly complete the sentence):

3 pts · Multiple choice · 4 alternatives

- [x] **require initialisation from another method**
- [ ] do not require initialisation from another method
- [x] **allow choosing large steps**
- [ ] can only be used with small steps

Feedback

Feedback when the question is answered correctly

Feedback when the question is answered partially correctly

Feedback when the question is answered incorrectly

This type of condition defines the value of the function (to be solved) at the boundary:

3 pts · Multiple choice · 4 alternatives

- ○ Neumann
- ◉ **Dirichlet**
- ○ Robin
- ○ Mixed

Feedback

Feedback when the question is answered correctly

Feedback when the question is answered partially correctly

Feedback when the question is answered incorrectly

Probability #43618152

30 pts · Last updated 3 Feb, 16:37

You are the designer of a bridge and want to study the traffic loads that it will face. You have measurements of axle loads in a nearby road that will connect to the bridge in the future.

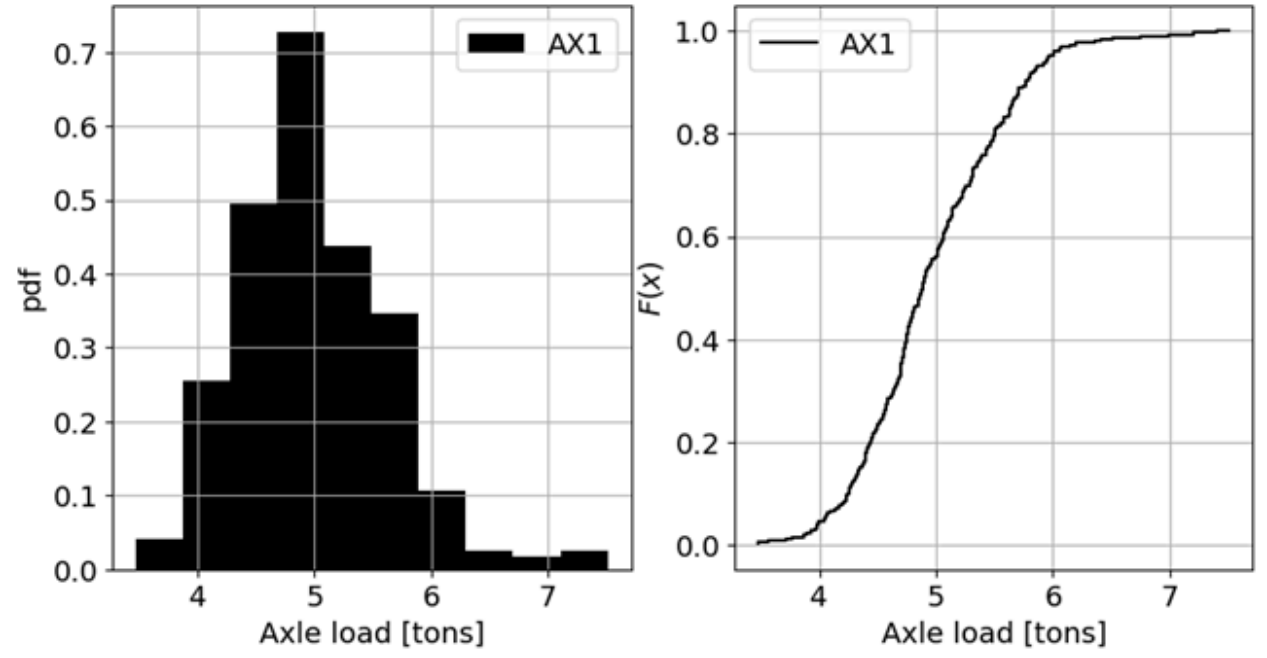
Axle loads are defined as the force transmitted by the wheels from the first axle to the ground in tons. You are focusing on trucks with 4 axles as they are the heavier vehicles in that road and, thus, will transmit higher axle loads to the bridge. Then, you have four axle loads per vehicle, denoted here as AX1, AX2, AX3 and AX4.

You start studying the first axle load, AX1. You plot the empirical CDF and PDF of the data, as shown in the image below.

Text

Picture1.png

Image 1/4 Page



What would be the most appropriate parametric distribution for AX1?

3 pts · Multiple choice · 3 alternatives

- (●) Gaussian
- ( ) Exponential
- ( ) Uniform

Feedback

Feedback when the question is answered correctly

Feedback when the question is answered partially correctly

Feedback when the question is answered incorrectly

Justify briefly your answer with **at least** one reason.

3 pts · Open · New page · 2/5 Page

Model answer

Gaussian because the pdf has a bell shape, approx. symmetric distribution. Exponential PDF would have an exponential decay having the maximum in 0.

Partial credit if Uniform is justified by saying that the CDF is approximately linear (not fully correct).

Gaussian, bell shape	3 pts
Uniform, linear CDF (Not fully correct)	1.5 pts
Some knowledge on the shape of some of the mentioned distributions	1 pt
Vague explanation, lack of reasons why specifically Gaussian is a good fit.	-1 pts
Small error such as uniform bounded in 0	-0.5 pts

Using the plot above, compute approximately $P(AX1 > 5)$.

3 pts · Open · 1/2 Page

Model answer

$P(AX1 > 5) = 1 - P(AX1 \leq 5) = 1 - 0.6 = 0.4$

Fully correct	3 pts
$P(AX1 > 5) = 1 - P(AX1 \leq 5)$	1.5 pts
Read from the adequate graph and finish computation	1.5 pts
Computation error	-0.5 pts

Another colleague is studying AX4. They have decided to fit a Gumbel distribution to the data of AX4. To assess the goodness of fit of the Gumbel distribution, they are using the Kolmogorov-Smirnov test.

Remember that the null hypothesis of the test is that the observations are coming from the fitted distribution and, thus, the tested distribution is a reasonable model for them. They have obtained a **statistic = 0.06** and a **p-value = 0.23** and asked for your help to interpret it considering a **significance level of 0.05**.

Is the Gumbel distribution a reasonable distribution for AX4 based on Kolmogorov-Smirnov test?
State yes or no, then briefly explain your reasoning using the quantitative results provided above.

6 pts · Open · 3/4 Page

Model answer

Yes, the p-value is the probability of the null hypothesis being true. If the p-value is higher than the significance, then the null hypothesis is accepted. Here 0.23 > 0.05, so we can accept that the observations come from a Gumbel distribution.

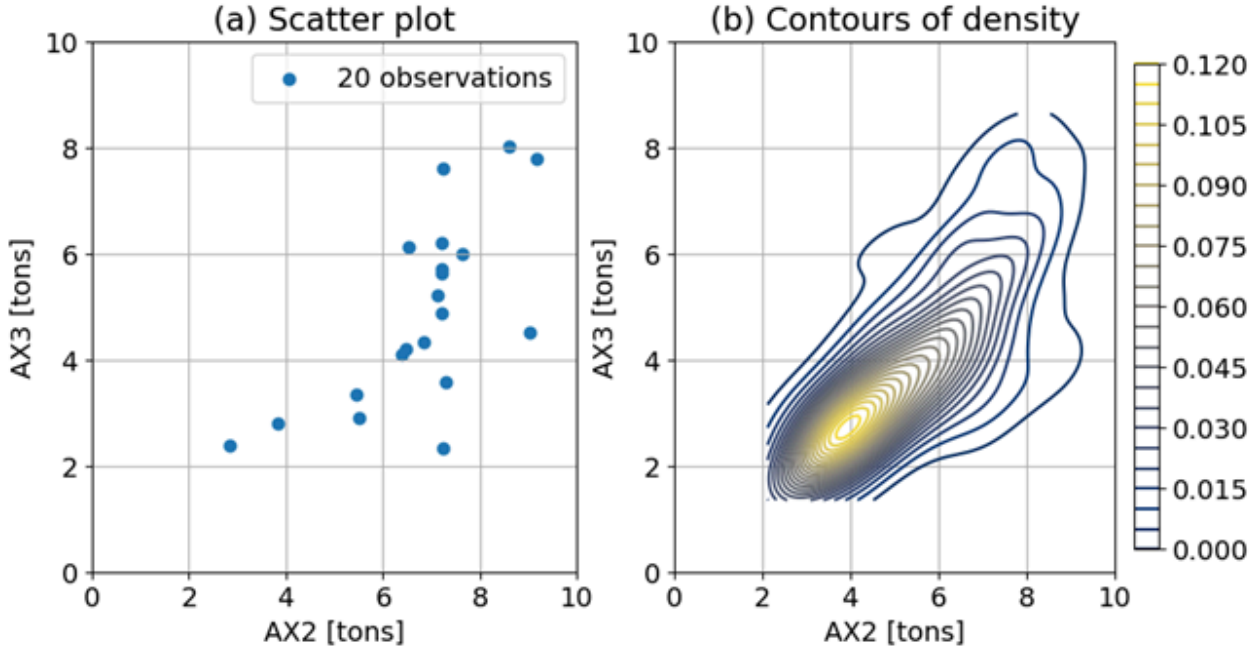
Fully correct	6 pts
Recognition of what an hypothesis test is (hypothesis, statistic, p-value) but wrong application.	2 pts

The multivariate probability team is working on the relationship between AX2 and AX3 and start by plotting some of the observations and the joint PDF as shown in the figure below.

Text

Picture2.png

Image 1/4 Page



Using the plots above, compute $P(AX2 \leq 6 | AX3 \leq 4)$

4 pts · Open · 1/2 Page

Model answer

$P(AX2 \leq 6 \mid AX3 \leq 4) = 4/6 = 0.67$

Using Bayes' theorem also possible

Fully correct	4 pts
Right numerator (mixing AND with conditional probability)	1 pt

Using the plots above, compute $P(AX2 > 8 \cap X3 > 6)$

4 pts · Open · 1/2 Page

Model answer

$P(AX2 > 8 \wedge AX3 > 6) = 2/20 = 0.1$

0 grade if you are doing the OR probability: 6/20

Fully correct	4 pts
Counting error/Minor error	-1 pts

The multivariate probability team has decided to use a multivariate Gaussian distribution to model the relationship between AX1, AX2, AX3 and AX4. They have obtained the following covariance matrix.

$$\Sigma = \begin{pmatrix} 0.40 & 0.76 & 0.37 & 0.35 \\ 0.76 & 3.60 & 2.30 & 2.01 \\ 0.37 & 2.30 & 2.4 & 2.21 \\ 0.35 & 2.01 & 2.21 & 2.30 \end{pmatrix}$$

Text

What is the variable with the highest variability?

3 pts · Multiple choice · 4 alternatives

Model answer

Higher values of the variance (diagonal of the covariance matrix) imply higher variability of the random variable. Note that all the variables have the same dimensions.

- ○ AX1
- ◉ AX2
- ○ AX3
- ○ AX4

Feedback

Feedback when the question is answered correctly

Feedback when the question is answered partially correctly

Feedback when the question is answered incorrectly

What is the pair of variables with the highest correlation?

4 pts · Multiple choice · 4 alternatives

Model answer

The off-diagonal values of the covariance matrix are the covariances. We can calculate the correlation as function of the covariance as

$\rho(AX1, AX2) = Cov(AX1, AX2)/(\sigma_1\sigma_2) = 0.76/(\sqrt{0.4}\sqrt{3.6}) = 0.63$
$\rho(AX2, AX3) = 2.3/(\sqrt{2.4}\sqrt{3.6}) = 0.78$
$\rho(AX1, AX4) = 0.35/(\sqrt{0.4}\sqrt{2.3}) = 0.36$
$\rho(AX3, AX4) = 2.21/(\sqrt{2.4}\sqrt{2.3}) = 0.94$

- ○ AX1-AX2
- ○ AX2-AX3
- ○ AX1-AX4
- ◉ AX3-AX4

Feedback

Feedback when the question is answered correctly

Feedback when the question is answered partially correctly

Feedback when the question is answered incorrectly

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Image 1 Page

Use the grid below if you run out of space for any exercise.
In that case, please indicate so at the original answer field.

The table below gives an overview of the questions to help you plan your time during the exam.

No.	Topic	Number of sub-parts	Points
1	Propagation of Uncertainty	4	20
2	Observation Theory	6	27
3	Numerical Modelling	5	23
4	Probability	8	30

Text

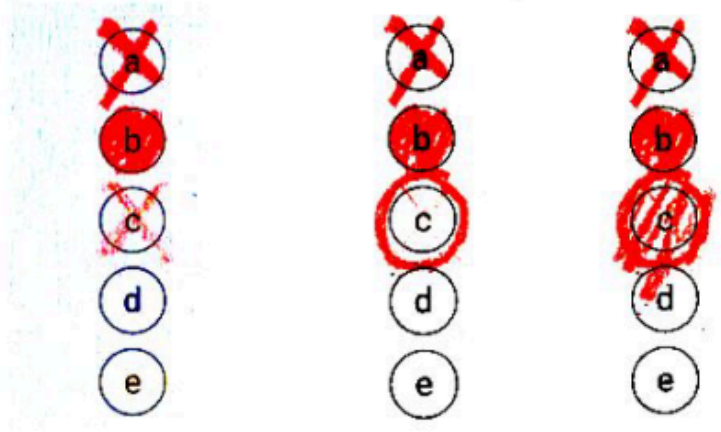
In case you want to correct your answer for a multiple choice question put an ARROW in front of your final answer. If you also make a mistake with your arrow, write a clear message on the page. Here are a few examples:

Text

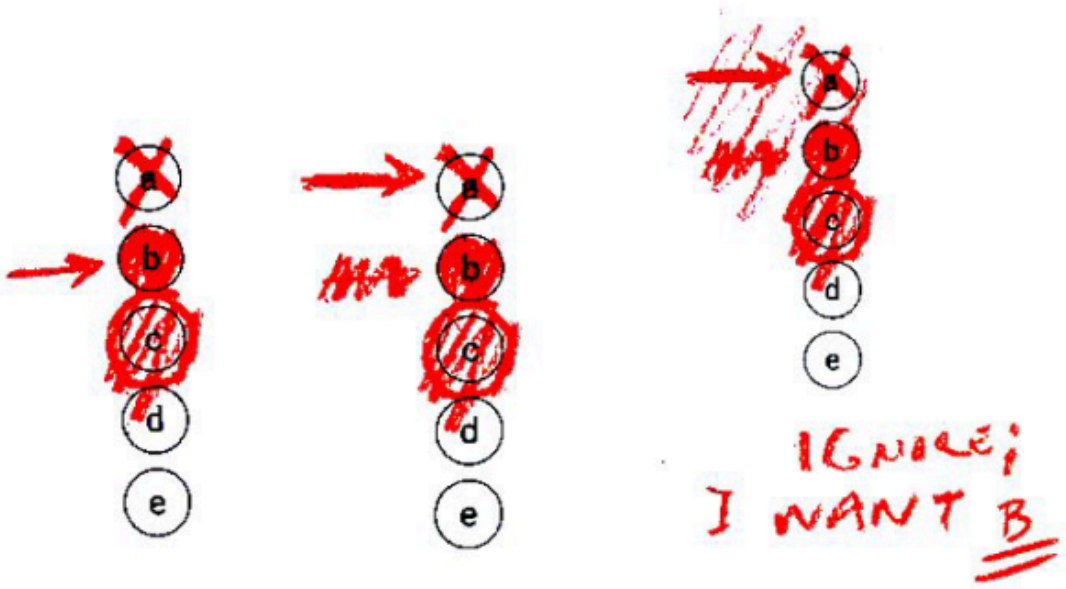
Screenshot 2025-01-19 223302.png

Image 2/5 Page

Examples of UNCLEAR multiple choice response:



Examples of CLEAR multiple choice response:



Answer: B Answer: A Answer: B

Multiple choice examples.